Finiteness spaces, étale groupoids and their convolution algebras

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Abstract

Given a ring R, we extend Ehrhard's linearization process by associating to any pre-finiteness space an R-module endowed with a Lefschetz topology. For a semigroup in the category of pre-finiteness spaces, one can endow this R-module with the convolution product to obtain an R-algebra.

As examples of pre-finiteness spaces, we study topological spaces with bounded subsets (i.e., included in a compact) taken to be the finitary subsets. We prove that we obtain a finiteness space from any *hemicompact* space via this construction. As a corollary, any étale Hausdorff groupoid induces a semigroup in pre-finiteness spaces and its associated convolution algebra is complete in the hemicompact case. This is in particular the case for the infinite paths groupoid associated to any countable row-finite directed graph.

Keywords: Finiteness space, internal semigroup, étale groupoid, linearization, Lefschetz topology, completion, row-finite directed graph.

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1 Introduction

In [2], the authors use finiteness spaces to construct new examples of convolution algebras. A *finiteness space* [3], defined in detail below, is a set equipped with two classes of subsets, the *finitary* and *cofinitary* subsets. These subsets have the property that the intersection of a finitary subset and a cofinitary subset must be finite. In fact, it is sufficient to identify just the finitary subsets since each class of subsets determine the other by an involutive duality $\mathcal{U} \mapsto \mathcal{U}^{\perp}$. So we will write $(X,\mathcal{U})$ for a finiteness space with $\mathcal{U}$ the class of finitary subsets. There are various categories with different choices of morphisms between finiteness spaces. But each of them has a monoidal structure. Thus we can consider internal semigroups in any of these categories. For a variety of reasons discussed in [2] (and recalled in Proposition 2.6 in the present paper), we consider here the category FinPf where morphisms are appropriate partial functions between finiteness spaces.

Given a finiteness space and a ring R, the *linearization process* (due to Ehrhard [3]) constructs a topological R-module $R\langle(X,\mathcal{U})\rangle$. The elements of this module are those functions from the finiteness space to the ring for which the support is finitary, i.e., an element of $\mathcal{U}$. The topology on $R\langle(X,\mathcal{U})\rangle$ is a *Lefschetz topology* as introduced in [6], where it is referred to as a linear topology. See also [1]. We extend this linearization process to the category $\mathsf{PreFinPf}$ of *pre-finiteness spaces*, a slightly more general notion, and show that the usual category of finiteness spaces is a reflective subcategory. We also show that the completion of the linearization of a pre-finiteness space is the linearization of its reflection in finiteness spaces.

Then, given an internal semigroup in $\mathsf{PreFinPf}$, we give $R\langle(X,\mathcal{U})\rangle$ an algebra structure using the usual convolution product. One can show this product is well-defined using the axioms of pre-finiteness spaces and of their morphisms. In this way, we are able to construct Ribenboim's rings of *generalized power series* [8, 9] as well as rings which had not been previously thought of as arising from convolution, such as the ring of *Puiseux series* (see [2]).

In the present paper, we consider this construction in the context of *étale groupoids*. In the classical theory, one associates to an étale groupoid $\mathcal{G}$ (with some restrictions) the convolution algebra of continuous functions $\mathcal{G}_1 \to \mathbb{C}$ vanishing outside a compact subset of $\mathcal{G}_1$. This of course bears more than a small resemblance to the situation in pre-finiteness spaces and it is this similarity we exploit in this paper.

On one hand, the étale groupoid case leads one to consider, for a topological space X, the pre-finiteness structure on X defined by the *bounded* subsets (i.e., the subsets included in a compact subset). This induces a functor

$$\mathrm{B}\colon \mathsf{LocFin} \to \mathsf{PreFinPf}$$

from the category of T_1 spaces and continuous, locally finite-to-one partial functions with closed domain. Thanks to that functor, one easily proves that each étale groupoid $\mathcal{G}$ with $\mathcal{G}_1$ Hausdorff induces a semigroup $(\mathcal{G}_1,\mathcal{U},m)$ in $\mathsf{PreFinPf}$ (with m the composition morphism, viewing it as a partial function) and thus a convolution R-algebra $R\langle(\mathcal{G}_1,\mathcal{U},m)\rangle$.

On the other hand, in the groupoid approach to C*-algebras [7, 11], the next step is to define a seminorm and then complete a quotient of the convolution algebra in this seminorm.

By analogy with our case, this leads us to consider the completion of the convolution algebra $R\langle(\mathcal{G}_1, \mathcal{U}, m)\rangle$ in the Lefschetz topology and to look when $R\langle(\mathcal{G}_1, \mathcal{U}, m)\rangle$ is already complete. By our completion theorem, the answer to the first question is simply that $R\langle(\mathcal{G}_1, \mathcal{U}^{\perp\perp}, m)\rangle$ is the completion of $R\langle(\mathcal{G}_1, \mathcal{U}, m)\rangle$; and thus $R\langle(\mathcal{G}_1, \mathcal{U}, m)\rangle$ is complete exactly when $(\mathcal{G}_1, \mathcal{U})$ is a finiteness space (if $R \neq 0$). We are thus looking for conditions on a topological space X to ensure that the bounded subsets pre-finiteness structure on X is a finiteness space. A sufficient but not necessary condition is that X is *hemicompact*. In particular, this is the case for σ*-locally compact spaces*. As a consequence, étale groupoids $\mathcal{G}$ with $\mathcal{G}_1$ hemicompact and Hausdorff induce a complete R-algebra $R\langle(\mathcal{G}_1, \mathcal{U}, m)\rangle$. This turns out to be a large class of groupoids as the example of shift equivalence on a countable row-finite directed graph as considered in [5] is an example of such an étale groupoid. It is in fact one of the fundamental examples in the groupoid approach to C*-algebras.

The organization of this paper is as follows. In the next section, we recall the basics of (pre-)finiteness spaces. In Section 3, we describe the process of linearization for a pre-finiteness space. We also consider the topological structure linearization induces and the convolution algebra given rise by a semigroup in pre-finiteness spaces. Section 4 is devoted to our completion theorem for the linearization of a pre-finiteness space. After briefly reviewing the basics of (étale) groupoids in Section 5, we then tackle the questions of knowing when continuous maps determine pre-finiteness maps between the bounded subsets pre-finiteness structures and when these structures are actually finiteness spaces. Finally, in Section 7, we consider the étale groupoid associated to shift equivalence for a countable row-finite directed graph and show that as a consequence of all that has gone before, this gives rise to a complete R-algebra.

Note: The rings we consider in this paper are associative but not necessarily commutative or unital.

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2 Pre-finiteness spaces

In order to recall Ehrhard's notion of *finiteness space* [3], we first recall the crucial *perp* definition:

Definition 2.1. Let X be a set and let $\mathcal{U}$ be a set of subsets of X, i.e., $\mathcal{U} \subseteq \mathcal{P}(X)$. Define $\mathcal{U}^\perp$ by:

$$\mathcal{U}^\perp = \{u' \subseteq X \,|\, \text{the set } u' \cap u \text{ is finite for all } u \in \mathcal{U}\}$$

It is immediate to check that one has $\mathcal{U} \subseteq \mathcal{U}^{\perp\perp}$ and $\mathcal{U}^{\perp\perp\perp} = \mathcal{U}^\perp$.

Following the ideas of [10], one defines:

Definition 2.2. A *pre-finiteness space* is a pair $(X,\mathcal{U})$ with X a set and $\mathcal{U} \subseteq \mathcal{P}(X)$ satisfying the following conditions:

- $\varnothing \in \mathcal{U}$;
- for each $x \in X$, $\{x\} \in \mathcal{U}$;
- if $u_1 \subseteq u_2 \in \mathcal{U}$, then $u_1 \in \mathcal{U}$;
- if $u_1, u_2 \in \mathcal{U}$, then $u_1 \cup u_2 \in \mathcal{U}$.

A *morphism of pre-finiteness spaces* $\alpha\colon (X,\mathcal{U}) \to (Y,\mathcal{V})$ is a partial function $\alpha\colon X \to Y$ such that

(1) for each $u \in \mathcal{U}$, $\alpha(u) \in \mathcal{V}$;

(2) for each $v' \in \mathcal{V}^{\perp}$, $\alpha^{-1}(v') \in \mathcal{U}^{\perp}$.

In presence of condition (1), condition (2) is equivalent to

(2') for each $y \in Y$, $\alpha^{-1}(y) \in \mathcal{U}^{\perp}$.

Under the classical composition of partial functions, pre-finiteness spaces and their morphisms form a category denoted by $\mathsf{PreFinPf}$.

The category $\mathsf{PreFinPf}$ is a symmetric monoidal category with unit $I = (\{*\}, \mathcal{P}(\{*\}))$ and tensor given by

$$(X,\mathcal{U}) \otimes (Y,\mathcal{V}) = (X \times Y, \{w \subseteq u \times v \,|\, u \in \mathcal{U},\, v \in \mathcal{V}\}).$$

The tensor of two morphisms and the unit, associativity and symmetry isomorphisms are defined in the obvious way.

Definition 2.3. A *finiteness space* is a pair $(X,\mathcal{U})$ with X a set and $\mathcal{U} \subseteq \mathcal{P}(X)$ such that $\mathcal{U}^{\perp\perp} = \mathcal{U}$. In particular, $(X,\mathcal{U})$ is a pre-finiteness space.

A *morphism of finiteness spaces* $f\colon (X,\mathcal{U}) \to (Y,\mathcal{V})$ is a morphism of pre-finiteness spaces. This forms the category FinPf which is a full subcategory of $\mathsf{PreFinPf}$.

Let us recall the following characterization from [3].

Proposition 2.4. *(Ehrhard) Let X be a set and $\mathcal{U} \subseteq \mathcal{P}(X)$ a downward closed set of subsets. For $u \subseteq X$, we have $u \in \mathcal{U}^{\perp\perp}$ if and only if, for any infinite subset v of u, there exists an infinite subset w of v such that $w \in \mathcal{U}$.*

The category FinPf is a symmetric monoidal category with unit $I = (\{*\}, \mathcal{P}(\{*\}))$ and tensor given by

$$\begin{aligned}(X,\mathcal{U}) \otimes (Y,\mathcal{V}) &= (X \times Y, \{w \subseteq u \times v \,|\, u \in \mathcal{U},\, v \in \mathcal{V}\}) \\ &= (X \times Y, \{u \times v \,|\, u \in \mathcal{U},\, v \in \mathcal{V}\}^{\perp\perp}).\end{aligned}$$

The tensor of two morphisms and the unit, associativity and symmetry isomorphisms are defined in the obvious way. With this definition, the inclusion functor $\mathsf{FinPf} \hookrightarrow \mathsf{PreFinPf}$ becomes strict symmetric monoidal, fully faithful and furthermore:

Proposition 2.5. *The inclusion functor* $I\colon \mathsf{FinPf} \hookrightarrow \mathsf{PreFinPf}$ *has a left adjoint which is also strict symmetric monoidal.*

Proof. We define

$$\begin{aligned} F = (-)^{\perp\perp}\colon \mathsf{PreFinPf} &\longrightarrow \mathsf{FinPf} \\ (X,\mathcal{U}) &\longmapsto (X,\mathcal{U}^{\perp\perp}) \\ \alpha &\longmapsto \alpha. \end{aligned}$$

Let us show that if $\alpha\colon (X,\mathcal{U}) \to (Y,\mathcal{V})$ is a morphism in $\mathsf{PreFinPf}$, then $\alpha\colon (X,\mathcal{U}^{\perp\perp}) \to (Y,\mathcal{V}^{\perp\perp})$ is a morphism in FinPf. Condition (2') being trivial, it suffices to prove condition (1). Let $u'' \in \mathcal{U}^{\perp\perp}$ and let us prove that $\alpha(u'') \in \mathcal{V}^{\perp\perp}$. So we consider $v' \in \mathcal{V}^{\perp}$ and we suppose by contradiction that $\alpha(u'') \cap v'$ is infinite. Thus there exists $(x_i \in u'')_{i\in\mathbb{N}}$ such that, for each $i \in \mathbb{N}$, $\alpha(x_i) \in v'$ and those $\alpha(x_i)$ are pairwise different. By Proposition 2.4, there exists an infinite subset $J \subseteq \mathbb{N}$ such that $\{x_i \,|\, i \in J\} \in \mathcal{U}$. But $\{\alpha(x_i) \,|\, i \in J\} = \alpha(\{x_i \,|\, i \in J\}) \cap v'$ is infinite, which leads to a contradiction since $\alpha(\{x_i \,|\, i \in J\}) \in \mathcal{V}$.

To show that F is a strict symmetric monoidal functor, we need to show that for prefiniteness spaces $(X,\mathcal{U})$ and $(Y,\mathcal{V})$,

$$(X,\mathcal{U}^{\perp\perp}) \otimes (Y,\mathcal{V}^{\perp\perp}) = (X \times Y, \{w \subseteq u'' \times v'' \,|\, u'' \in \mathcal{U}^{\perp\perp},\, v'' \in \mathcal{V}^{\perp\perp}\})$$

is equal to

$$F((X,\mathcal{U}) \otimes (Y,\mathcal{V})) = (X \times Y, \{w \subseteq u \times v \,|\, u \in \mathcal{U},\, v \in \mathcal{V}\}^{\perp\perp}).$$

Since $\{w \subseteq u \times v \,|\, u \in \mathcal{U},\, v \in \mathcal{V}\} \subseteq \{w \subseteq u'' \times v'' \,|\, u'' \in \mathcal{U}^{\perp\perp},\, v'' \in \mathcal{V}^{\perp\perp}\}$ and $(X,\mathcal{U}^{\perp\perp}) \otimes (Y,\mathcal{V}^{\perp\perp})$ is a finiteness space, we already have

$$\{w \subseteq u \times v \,|\, u \in \mathcal{U},\, v \in \mathcal{V}\}^{\perp\perp} \subseteq \{w \subseteq u'' \times v'' \,|\, u'' \in \mathcal{U}^{\perp\perp},\, v'' \in \mathcal{V}^{\perp\perp}\}.$$

For the reverse inclusion, we consider $w \subseteq u'' \times v''$ with $u'' \in \mathcal{U}^{\perp\perp}$ and $v'' \in \mathcal{V}^{\perp\perp}$. If $\{(x_i, y_i) \,|\, i \in I\}$ is an infinite subset of w, by Proposition 2.4, there exists an infinite subset $J \subseteq I$ such that $\{x_i \,|\, i \in J\} \in \mathcal{U}$. Again by Proposition 2.4, there exists an infinite subset $K \subseteq J$ such that $\{y_i \,|\, i \in K\} \in \mathcal{V}$. Therefore $\{(x_i, y_i) \,|\, i \in K\} \in \{z \subseteq u \times v \,|\, u \in \mathcal{U},\, v \in \mathcal{V}\}$, which by Proposition 2.4, means that $w \in \{z \subseteq u \times v \,|\, u \in \mathcal{U},\, v \in \mathcal{V}\}^{\perp\perp}$. This shows that $F\colon \mathsf{PreFinPf} \to \mathsf{FinPf}$ is a strict symmetric monoidal functor. We have a symmetric monoidal adjunction

$$\begin{tikzcd} \mathsf{FinPf} \arrow[r, hook, "I"', bend right=0] \arrow[r, phantom, "\perp"] & \mathsf{PreFinPf} \arrow[l, bend right, "F=(-)^{\perp\perp}"'] \end{tikzcd}$$

where the unit $\eta\colon 1_{\mathsf{PreFinPf}} \Rightarrow IF$ and the counit $\varepsilon\colon FI \Rightarrow 1_{\mathsf{FinPf}}$ are the monoidal natural transformations given respectively by

$$\eta_{(X,\mathcal{U})} = 1_X\colon (X,\mathcal{U}) \to (X,\mathcal{U}^{\perp\perp})$$

and ε is the identity on 1_{FinPf}. $\square$

For a monoidal category $\mathbb{M}$, we denote by $\mathrm{SG}(\mathbb{M})$ the category of internal semigroups in $\mathbb{M}$ (i.e., objects M equipped with an associative map $m\colon M \otimes M \to M$). The above monoidal adjunction gives rise to the adjunction:

$$\begin{tikzcd} \mathrm{SG}(\mathsf{FinPf}) \arrow[r, hook, bend right, ""{name=D}] & \mathrm{SG}(\mathsf{PreFinPf}) \arrow[l, bend right, "(-)^{\perp\perp}"', ""{name=U}] \arrow[from=U, to=D, phantom, "\perp"] \end{tikzcd}$$

The category FinPf has the following additional properties [2].

Proposition 2.6. *The category* FinPf *is a symmetric monoidal closed category. It is moreover pointed, complete and cocomplete.*

Let us conclude this section with a technical lemma we will need in Section 6.

Lemma 2.7. *Let* $(X,\mathcal{U})$ *be a pre-finiteness space which admits a countable family* $(u_i \in \mathcal{U})_{i\in\mathbb{N}}$ *such that for each* $u \in \mathcal{U}$*, there exists* $i \in \mathbb{N}$ *with* $u \subseteq u_i$*. Then* $(X,\mathcal{U})$ *is a finiteness space.*

Proof. Up to replace $u_0, u_1, u_2, \ldots$ by $u_0, u_0 \cup u_1, u_0 \cup u_1 \cup u_2, \ldots$, we can assume without loss of generality that

$$u_0 \subseteq u_1 \subseteq u_2 \subseteq \cdots$$

By contradiction, suppose we have $u'' \in \mathcal{U}^{\perp\perp} \setminus \mathcal{U}$. Since $u'' \notin \mathcal{U}$, for each $i \in \mathbb{N}$, $u'' \nsubseteq u_i$. By the axiom of choice, we choose for each $i \in \mathbb{N}$ a $x_i \in u'' \setminus u_i$. If the set $v = \{x_i \,|\, i \in \mathbb{N}\}$ is finite, there would exist $x \in v$ such that $x \notin \bigcup_{i\in\mathbb{N}} u_i$. But the singleton $\{x\}$ is in $\mathcal{U}$ which contradicts our assumptions. Thus $v \subseteq u''$ is infinite. By Proposition 2.4, there exists an infinite set $u \subseteq v$ with $u \in \mathcal{U}$. By assumption, there exists $i \in \mathbb{N}$ such that $u \subseteq u_i$, contradicting the construction of v. □

3 Linearization

The notion of linear topology used in this and the following sections is due to Lefschetz [6]. Let R be a ring (not necessarily commutative or unital). A *linearly Hausdorff R-module* is a left R-module M together with a Hausdorff topology on M which is invariant by translations and admits a neighbourhood basis of 0 consisting of submodules of M. In particular, $+\colon M^2 \to M$, $-\colon M \to M$ and the multiplication $R \times M \to M$ are continuous functions when R is considered with the discrete topology. Linearly Hausdorff R-modules and continuous left R-module morphisms form the category $\mathsf{Haus}\text{-}R\text{-}\mathsf{Mod}$.

Following the ideas of [3], we now define the linearization functor

$$R\langle - \rangle\colon\ \mathsf{PreFinPf} \to \mathsf{Haus}\text{-}R\text{-}\mathsf{Mod}\,.$$

Given a pre-finiteness space $(X,\mathcal{U})$, we define $R\langle (X,\mathcal{U}) \rangle$ as the left R-module

$$R\langle (X,\mathcal{U}) \rangle = \{ f\colon X \to R \,|\, |f| \in \mathcal{U} \}$$

where $|f|$ is the support of f: $|f| = \mathrm{supp}(f) = \{x \in X \mid f(x) \neq 0\}$. The module operations are defined componentwise. It is straightforward to prove that the result of these operations satisfy the condition on the support using the axioms of Definition 2.2.

For $u' \in \mathcal{U}^\perp$, we set $V_{u'}$ to be the submodule

$$V_{u'} = \{f \in R\langle(X,\mathcal{U})\rangle \mid f_{|_{u'}} = 0\}.$$

One then says that $V \subseteq R\langle(X,\mathcal{U})\rangle$ is *open* if and only if for any $g \in V$, there exists $u' \in \mathcal{U}^\perp$ such that $g + V_{u'} \subseteq V$. This defines a topology on $R\langle(X,\mathcal{U})\rangle$ for which $g + V_{u'}$ is open for any $g \in R\langle(X,\mathcal{U})\rangle$ and any $u' \in \mathcal{U}^\perp$. This topology is Hausdorff since given $f \neq g \in R\langle(X,\mathcal{U})\rangle$, the opens $f + V_{\{x\}}$ and $g + V_{\{x\}}$ separate f and g for any x such that $f(x) \neq g(x)$. This topology is clearly invariant by translations and $\{V_{u'} \mid u' \in \mathcal{U}^\perp\}$ is a neighbourhood basis of 0. Therefore $R\langle(X,\mathcal{U})\rangle$ is a linear Hausdorff R-module.

Given a morphism $\alpha\colon (X,\mathcal{U}) \to (Y,\mathcal{V})$ in $\mathsf{PreFinPf}$, one defines $R\langle\alpha\rangle\colon R\langle(X,\mathcal{U})\rangle \to R\langle(Y,\mathcal{V})\rangle$ via

$$R\langle\alpha\rangle(f)(y) = \sum_{x \in \alpha^{-1}(y) \cap |f|} f(x)$$

for any $f \in R\langle(X,\mathcal{U})\rangle$ and any $y \in Y$. To see it is continuous, it suffices to notice that for $h \in R\langle(Y,\mathcal{V})\rangle$, $v' \in \mathcal{V}^\perp$ and $f \in R\langle\alpha\rangle^{-1}(h + V_{v'})$, we have $f + V_{\alpha^{-1}(v')} \subseteq R\langle\alpha\rangle^{-1}(h + V_{v'})$. This completes the definition of the linearization functor $R\langle-\rangle\colon \mathsf{PreFinPf} \to \mathsf{Haus}\text{-}R\text{-}\mathsf{Mod}$.

For a non-commutative ring, the notion of an algebra not being standard, we use the following one here. An *R-algebra* is given by a ring $(A, +, 0, -, \cdot)$ together with actions $R \times A \to A$ and $A \times R \to A$ making A an R-bimodule and satisfying the axioms

- $r(ab) = (ra)b$;
- $(ab)r = a(br)$;
- $(ar)b = a(rb)$

for any $a, b \in A$ and any $r \in R$. A *linearly Hausdorff R-algebra* is an R-algebra A together with a Hausdorff topology on A which is invariant by translations and admits a neighbourhood basis of 0 consisting of sub-bimodules of A. Linearly Hausdorff R-algebras and continuous R-algebra morphisms form the category $\mathsf{Haus}\text{-}R\text{-}\mathsf{Alg}$.

We can extend the linearization functor $R\langle-\rangle\colon \mathsf{PreFinPf} \to \mathsf{Haus}\text{-}R\text{-}\mathsf{Mod}$ to

$$R\langle-\rangle\colon \mathrm{SG}(\mathsf{PreFinPf}) \to \mathsf{Haus}\text{-}R\text{-}\mathsf{Alg}$$

as follows. Given a semigroup $\mathbb{X}$ in $\mathsf{PreFinPf}$, i.e., a pre-finiteness space $(X,\mathcal{U})$ equipped with a semigroup law $m\colon (X,\mathcal{U}) \otimes (X,\mathcal{U}) \to (X,\mathcal{U})$, we define $R\langle\mathbb{X}\rangle$ as $R\langle(X,\mathcal{U})\rangle$ together with the componentwise bimodule operations and the multiplication $\cdot\colon R\langle\mathbb{X}\rangle \times R\langle\mathbb{X}\rangle \to R\langle\mathbb{X}\rangle$ given by the convolution product

$$(f \cdot g)(x) = \sum_{\substack{(y,z) \in m^{-1}(x) \\ y \in |f| \\ z \in |g|}} f(y)g(z)$$

for $f, g \in R\langle\mathbb{X}\rangle$ and $x \in X$. This is a finite sum since m satisfies condition (2'). Moreover, $|f \cdot g| \subseteq m(|f| \times |g|) \in \mathcal{U}$ since m satisfies condition (1). For a semigroup homomorphism $\alpha\colon \mathbb{X} = (X, \mathcal{U}, m) \to \mathbb{Y} = (Y, \mathcal{V}, n)$ in $\mathsf{PreFinPf}$, the map $R\langle\alpha\rangle\colon R\langle\mathbb{X}\rangle \to R\langle\mathbb{Y}\rangle$ is defined as previously. It preserves the multiplication since

$$\begin{aligned}
R\langle\alpha\rangle(f \cdot g)(y) &= \sum_{x \in \alpha^{-1}(y) \cap |f \cdot g|} \sum_{\substack{(x_1,x_2) \in m^{-1}(x) \\ x_1 \in |f| \\ x_2 \in |g|}} f(x_1)g(x_2) \\
&= \sum_{\substack{(x_1,x_2) \in |f| \times |g| \\ \alpha(m(x_1,x_2)) = y}} f(x_1)g(x_2) \\
&= \sum_{\substack{(x_1,x_2) \in |f| \times |g| \\ n(\alpha(x_1),\alpha(x_2)) = y}} f(x_1)g(x_2) \\
&= \sum_{\substack{(y_1,y_2) \in n^{-1}(y) \\ y_1 \in \alpha(|f|) \\ y_2 \in \alpha(|g|)}} \left(\sum_{x_1 \in \alpha^{-1}(y_1) \cap |f|} f(x_1) \right) \cdot \left(\sum_{x_2 \in \alpha^{-1}(y_2) \cap |g|} g(x_2) \right) \\
&= (R\langle\alpha\rangle(f) \cdot R\langle\alpha\rangle(g))(y)
\end{aligned}$$

for $f, g \in R\langle\mathbb{X}\rangle$ and $y \in Y$.

4 The completion theorem

Let R be a ring. Each linearly Hausdorff R-module M is a uniform space when we say that $W \subseteq M \times M$ is an *entourage* if and only if there exists a neighbourhood V of 0 such that for $x, y \in M$ with $x - y \in V$, we have $(x, y) \in W$. For a primer on the theory of uniform spaces, see [4]. This induces a forgetful functor $\mathsf{Haus}\text{-}R\text{-}\mathsf{Mod} \to \mathsf{Unif}$ to the category of uniform spaces and uniformly continuous functions. That way, we can consider Cauchy nets in any linearly Hausdorff R-module and the notion of completeness makes sense.

Theorem 4.1. *Let $R \neq 0$ be a ring and $(X,\mathcal{U})$ a pre-finiteness space. Then $R\langle(X,\mathcal{U})\rangle$ is complete if and only if $(X,\mathcal{U})$ is a finiteness space. Moreover, $R\langle(X,\mathcal{U}^{\perp\perp})\rangle$ is the completion of $R\langle(X,\mathcal{U})\rangle$, i.e., for any uniformly continuous function $h\colon R\langle(X,\mathcal{U})\rangle \to Z$ to a complete uniform Hausdorff space, there exists a unique uniformly continuous function $\overline{h}\colon R\langle(X,\mathcal{U}^{\perp\perp})\rangle \to Z$ such that $\overline{h} \circ R\langle\eta_{(X,\mathcal{U})}\rangle = h$.*

$$\begin{tikzcd}
R\langle(X,\mathcal{U})\rangle \arrow[r, hook, "R\langle\eta_{(X,\mathcal{U})}\rangle"] \arrow[dr, "\forall h"'] & R\langle(X,\mathcal{U}^{\perp\perp})\rangle \arrow[d, dotted, "\exists!\,\overline{h}"] \\
& Z
\end{tikzcd}$$

Proof. The fact that $R\langle(X,\mathcal{U})\rangle$ is complete if $(X,\mathcal{U})$ is a finiteness space is due to Ehrhard [3]. Since his proof only includes the case of sequences, let us prove it in full generality here. So let D be a directed set and $a\colon D \to R\langle(X,\mathcal{U})\rangle$ a Cauchy net in $R\langle(X,\mathcal{U})\rangle$. For $x \in X$, we know that $\{x\} \in \mathcal{U}^\perp$, so $V_{\{x\}}$ is an open neighbourhood of 0. Thus, there exists $N_x \in D$ such that if $m,n \geqslant N_x$, then $a_m - a_n \in V_{\{x\}}$, i.e., $a_m(x) = a_n(x)$. We define $f\colon X \to R$ by $f(x) = a_{N_x}(x)$ for $x \in X$. Notice that this definition is independent of the choice of N_x since D is a directed set. For $u' \in \mathcal{U}^\perp$, since a is a Cauchy net and $V_{u'}$ is an open neighbourhood of 0, we know there exists $N_{u'} \in D$ such that $a_{m|_{u'}} = a_{n|_{u'}}$ for any $m,n \geqslant N_{u'}$. Thus, for any $n \geqslant N_{u'}$ and any $x \in u'$, consider a $N' \in D$ with $N' \geqslant N_{u'}$ and $N' \geqslant N_x$. We have $a_n(x) = a_{N'}(x) = a_{N_x}(x) = f(x)$. This shows that $a_{n|_{u'}} = f_{|_{u'}}$ for any $n \geqslant N_{u'}$. We deduce that $|f| \in \mathcal{U}^{\perp\perp} = \mathcal{U}$ since $|f| \cap u' = |a_{N_{u'}}| \cap u'$ for any $u' \in \mathcal{U}^\perp$. This also proves that the net a converges to f. Hence $R\langle(X,\mathcal{U})\rangle$ is complete.

Let us now prove the universal property. We first notice that $R\langle\eta_{(X,\mathcal{U})}\rangle\colon R\langle(X,\mathcal{U})\rangle \hookrightarrow R\langle(X,\mathcal{U}^{\perp\perp})\rangle$ is the natural inclusion. Let us consider a complete uniform Hausdorff space Z and a uniformly continuous function $h\colon R\langle(X,\mathcal{U})\rangle \to Z$. We first notice that $D = \mathcal{U}$ is a directed set when ordered by inclusion. Let also $f \in R\langle(X,\mathcal{U}^{\perp\perp})\rangle$. For $u \in \mathcal{U}$, we set

$$a_u(x) = \begin{cases} f(x) & \text{if } x \in u \\ 0 & \text{otherwise.} \end{cases}$$

Thus $a_u \in R\langle(X,\mathcal{U})\rangle$. Let us prove that $a\colon \mathcal{U} \to R\langle(X,\mathcal{U})\rangle$ is a Cauchy net: For each neighbourhood $V_{u'}$ of 0 where $u' \in \mathcal{U}^\perp$, we know that $u' \cap |f| \in \mathcal{U}$ since it is finite. Thus, for each $u \in \mathcal{U}$ such that $u \supseteq u' \cap |f|$, $a_{u|_{u'}} = f_{|_{u'}} = a_{u'\cap|f||_{u'}}$, i.e., $a_u - a_{u'\cap|f|} \in V_{u'}$ and a is indeed a Cauchy net. The same argument also shows that a converges to f in $R\langle(X,\mathcal{U}^{\perp\perp})\rangle$. These facts prove that $(h(a_u))_{u\in\mathcal{U}}$ is a Cauchy net in Z which must converges by completeness of Z. Moreover, we must define $\overline{h}(f)$ as the limit of $(h(a_u))_{u\in\mathcal{U}}$. This already shows that $\overline{h}$ is unique since Z is Hausdorff. Moreover, if $f \in R\langle(X,\mathcal{U})\rangle$, a converges to f in $R\langle(X,\mathcal{U})\rangle$ and thus $\overline{h}(f) = h(f)$. It remains to prove that $\overline{h}$ is uniformly continuous. Let W_1 be an entourage of Z and consider W_2 another entourage of Z such that $W_2 \circ W_2 \circ W_2 \subseteq W_1$. Since h is uniformly continuous, $(h \times h)^{-1}(W_2)$ is an entourage of $R\langle(X,\mathcal{U})\rangle$. So, there exists $u' \in \mathcal{U}$ such that for $f, f' \in R\langle(X,\mathcal{U})\rangle$ satisfying $f_{|_{u'}} = f'_{|_{u'}}$, we have $(h(f), h(f')) \in W_2$. To prove that $(\overline{h} \times \overline{h})^{-1}(W_1)$ is an entourage of $R\langle(X,\mathcal{U}^{\perp\perp})\rangle$, it suffices to show that for any $f, f' \in R\langle(X,\mathcal{U}^{\perp\perp})\rangle$ satisfying $f_{|_{u'}} = f'_{|_{u'}}$, then $(\overline{h}(f), \overline{h}(f')) \in W_1$. Let us consider such f and f' with a and a' the Cauchy nets defined above which converge to f and f' respectively. Since $(h(a_u))_{u\in\mathcal{U}}$ converges to $\overline{h}(f)$, there exists $u_1 \in \mathcal{U}$ such that if $u \in \mathcal{U}$ satisfies $u \supseteq u_1$, then $(\overline{h}(f), h(a_u)) \in W_2$. Similarly, there exists $u_2 \in \mathcal{U}$ such that for any $u \in \mathcal{U}$ with $u \supseteq u_2$, we have $(h(a'_u), \overline{h}(f')) \in W_2$. Moreover, for any $u \in \mathcal{U}$ such that $u \supseteq u' \cap (|f| \cup |f'|)$, we have $a_{u|_{u'}} = f_{|_{u'}} = f'_{|_{u'}} = a'_{u|_{u'}}$, which implies that $(h(a_u), h(a'_u)) \in W_2$. Considering any $u \in \mathcal{U}$ such that $u \supseteq u_1 \cup u_2 \cup (u' \cap (|f| \cup |f'|))$, we have $(\overline{h}(f), h(a_u)) \in W_2$, $(h(a_u), h(a'_u)) \in W_2$ and $(h(a'_u), \overline{h}(f')) \in W_2$. Thus $(\overline{h}(f), \overline{h}(f')) \in W_2 \circ W_2 \circ W_2 \subseteq W_1$, concluding the proof that $\overline{h}$ is uniformly continuous.

Finally, if $R\langle(X,\mathcal{U})\rangle$ is complete, this universal property implies that $R\langle(X,\mathcal{U})\rangle =$

$R\langle(X,\mathcal{U}^{\perp\perp})\rangle$. Since $R \neq 0$, we can consider $r \neq 0 \in R$. For any $u'' \in \mathcal{U}^{\perp\perp}$, the support of the map $r_{u''}$ defined by

$$r_{u''}(x) = \begin{cases} r & \text{if } x \in u'' \\ 0 & \text{otherwise} \end{cases}$$

is u''. So $r_{u''} \in R\langle(X,\mathcal{U}^{\perp\perp})\rangle = R\langle(X,\mathcal{U})\rangle$. This means that $u'' \in \mathcal{U}$ and thus $\mathcal{U}^{\perp\perp} \subseteq \mathcal{U}$, proving that $(X,\mathcal{U})$ is a finiteness space. $\square$

5 Étale groupoids and their convolution algebras

We recommend the reference [11] for the following section. Let us first recall some well-known categorical notions.

Definition 5.1. A *groupoid* is a (small) category in which every morphism is invertible.

That is, a groupoid $\mathcal{G}$ is a pair of sets $\mathcal{G}_1$ (arrows) and $\mathcal{G}_0$ (objects) together with morphisms

- $d, r \colon \mathcal{G}_1 \rightrightarrows \mathcal{G}_0$ (domain and range)
- $m \colon \{(\alpha,\beta) \in \mathcal{G}_1 \times \mathcal{G}_1 \mid r(\alpha) = d(\beta)\} \to \mathcal{G}_1$ (composition, or partial multiplication)
- $u \colon \mathcal{G}_0 \to \mathcal{G}_1$ (unit)
- $i \colon \mathcal{G}_1 \to \mathcal{G}_1$ (inverse)

satisfying the appropriate axioms. This can be summarized in the following diagram:

$$\begin{tikzcd} \mathcal{G}_1 \times_{\mathcal{G}_0} \mathcal{G}_1 \arrow[r, "m"] & \mathcal{G}_1 \arrow[loop above, "i"] \arrow[r, shift left=2, "d"] \arrow[r, shift right=2, "r"'] & \mathcal{G}_0 \arrow[l, "u"] \end{tikzcd}$$

where $\mathcal{G}_1 \times_{\mathcal{G}_0} \mathcal{G}_1$ is the pullback of d along r.

If this diagram is considered internally to a category $\mathcal{C}$, one then calls it an *internal groupoid in* $\mathcal{C}$. For instance, one has:

- A *topological groupoid* is an internal groupoid in Top, the category of topological spaces and continuous maps. Thus, a topological groupoid is a groupoid $\mathcal{G}$ where $\mathcal{G}_0$ and $\mathcal{G}_1$ are endowed with a topological structure such that d, r, m, u and i are continuous (with the topology on $\mathcal{G}_1 \times_{\mathcal{G}_0} \mathcal{G}_1$ induced by the product topology on $\mathcal{G}_1 \times \mathcal{G}_1$).
- An *étale groupoid* is an internal groupoid in the category of topological spaces and local homeomorphisms. Equivalently, it is a topological groupoid for which the map $ur \colon \mathcal{G}_1 \to \mathcal{G}_1$ is a local homeomorphism, and consequently so are all the structure maps d, r, m, u and i.

In the reference [11], one finds the following Proposition 3.1.1, where as usual, for a topological space X,

$$\mathcal{C}_c(X) = \{\text{continuous } f\colon X \to \mathbb{C} \mid \operatorname{supp}(f) \text{ is included in a compact subset of } X\}.$$

Proposition 5.2. *[11] Let $\mathcal{G}$ be an étale groupoid for which $\mathcal{G}_1$ is second-countable locally compact and Hausdorff. For $f, g \in \mathcal{C}_c(\mathcal{G}_1)$ and $\gamma \in \mathcal{G}_1$, the set*

$$\{(\alpha, \beta) \in \mathcal{G}_1 \times_{\mathcal{G}_0} \mathcal{G}_1 \mid m(\alpha, \beta) = \gamma \text{ and } f(\alpha)g(\beta) \neq 0\}$$

is finite. The complex vector space $\mathcal{C}_c(\mathcal{G}_1)$ is a $*$-algebra with multiplication given by

$$(f \cdot g)(\gamma) = \sum_{\substack{(\alpha,\beta) \mid m(\alpha,\beta)=\gamma \\ f(\alpha)g(\beta)\neq 0}} f(\alpha)g(\beta).$$

The key observation is the finiteness of the above set. This is derived by observing that this set is the intersection of a compact set and a closed, discrete set and so must be finite.

6 Topological spaces as pre-finiteness spaces

The obvious relationship between Proposition 5.2 and the linearization process of Section 3 led the authors to consider, for a topological space X, the pre-finiteness structure of subsets of X included in a compact set. In this section, we are going to investigate this pre-finiteness structure on topological spaces, and in particular in which sense it is functorial. In addition, Theorem 3.2.2 in [11] constructs a C*-algebra from an étale groupoid $\mathcal{G}$ as in Proposition 5.2 using a completion process from $\mathcal{C}_c(\mathcal{G}_1)$. This led the authors to consider the questions of describing the completion of the linearization of this particular pre-finiteness structure on X in the Lefschetz topology and to have conditions on X to ensure its linearization to be complete. While the former question is answered by Theorem 4.1, the latter is treated at the end of this section.

Definition 6.1. Let X be a topological space. A subset $u \subseteq X$ is said to be *bounded* if it is included in a compact subset of X. We denote by $\mathcal{B}_X$ the set of bounded subsets of X.

If X is a Hausdorff space, a subset $u \subseteq X$ is bounded if and only if it is relatively compact, i.e., its closure $\overline{u}$ is compact. As expected, we have the following immediate proposition.

Proposition 6.2. *If X is a topological space, then $(X, \mathcal{B}_X)$ forms a pre-finiteness space.*

We now investigate how to turn this construction to a functor.

Definition 6.3. Let $f\colon X \to Y$ be a partial function between two topological spaces and denote by $\operatorname{dom}(f)$ its domain endowed with the induced topology from X. We say that f is *continuous* when the (total) function $f\colon \operatorname{dom}(f) \to Y$ is continuous. We say that f is *locally finite-to-one* when the (total) function $f\colon \operatorname{dom}(f) \to Y$ is locally finite-to-one, i.e., for each $x \in \operatorname{dom}(f)$, there exists a neighbourhood U of x such that the restriction map $f_{|_U}$ has no infinite fibres.

Proposition 6.4. *Let $f\colon X \to Y$ be a continuous locally finite-to-one partial function between two topological spaces such that Y is T_1 and* $\mathrm{dom}(f)$ *is closed in X. Then f induces a morphism of pre-finiteness spaces $f\colon (X, \mathcal{B}_X) \to (Y, \mathcal{B}_Y)$.*

Proof. If K is a compact subset of X, $K \cap \mathrm{dom}(f)$ is a compact subset of $\mathrm{dom}(f)$ since $\mathrm{dom}(f)$ is closed. Therefore, $f(K) = f(K \cap \mathrm{dom}(f))$ is compact in Y since $f\colon \mathrm{dom}(f) \to Y$ is a (total) continuous function. This already proves condition (1) of the definition of morphisms in $\mathsf{PreFinPf}$ for f.

To prove condition (2'), let $y \in Y$, K a compact subset of X and let us prove that $f^{-1}(y) \cap K$ is finite. For each $x \in f^{-1}(y) \cap K$, there exists an open $U_x \subseteq X$ such that $x \in U_x$ and $f_{|U_x}$ is finite-to-one. Moreover, for each $x' \in (K \cap \mathrm{dom}(f)) \setminus f^{-1}(y)$, since $f(x') \neq y$ and Y is T_1, there exists an open $V_{x'} \subseteq Y$ such that $f(x') \in V_{x'}$ but $y \notin V_{x'}$. Thus, there exists an open $W_{x'} \subseteq X$ such that $x' \in W_{x'}$ and $W_{x'} \cap \mathrm{dom}(f) = f^{-1}(V_{x'})$. That way, we have constructed an open cover of K:

$$K \subseteq \left(\bigcup_{x \in f^{-1}(y) \cap K} U_x \right) \cup \left(\bigcup_{x' \in (K \cap \mathrm{dom}(f)) \setminus f^{-1}(y)} W_{x'} \right) \cup (\mathrm{dom}(f))^{\mathrm{C}}$$

Notice that $f^{-1}(y) \cap W_{x'} = \varnothing$ for any $x' \in (K \cap \mathrm{dom}(f)) \setminus f^{-1}(y)$ and $f^{-1}(y) \cap (\mathrm{dom}(f))^{\mathrm{C}} = \varnothing$. Since K is compact, it admits a finite subcover. If $f^{-1}(y) \cap K$ is infinite, this would imply the existence of $x \in f^{-1}(y) \cap K$ such that U_x contains infinitely many elements of $f^{-1}(y) \cap K$. However, this is impossible since $f_{|U_x}$ is finite-to-one. $\square$

Example 6.5. Let $U \subseteq \mathbb{C}$ be a connected open subset of the complex plane. By the 'principle of isolated zeroes', any non-constant analytic function $f\colon U \to \mathbb{C}$ induces a morphism of pre-finiteness spaces $(U, \mathcal{B}_U) \to (\mathbb{C}, \mathcal{B}_{\mathbb{C}})$. The same holds if we replace $\mathbb{C}$ by the real line $\mathbb{R}$.

We have some kind of converse of Proposition 6.4.

Proposition 6.6. *Let $f\colon X \to Y$ be a partial function between topological spaces where X is locally compact. If f induces a morphism of pre-finiteness spaces $f\colon (X, \mathcal{B}_X) \to (Y, \mathcal{B}_Y)$, then f is locally finite-to-one.*

Proof. If f is not locally finite-to-one, then there is some point $x \in \mathrm{dom}(f)$ such that for any neighbourhood $\mathrm{dom}(f) \supseteq U \ni x$, there is some $y \in Y$ such that $U \cap f^{-1}(y)$ is infinite. Since this is true for any neighbourhood U of x in $\mathrm{dom}(f)$, applying local compactness of X, we get that there is a compact $K \subseteq X$ such that $K \cap f^{-1}(y)$ is infinite. But this contradicts the condition that $f^{-1}(y) \in \mathcal{B}_X^{\perp}$. $\square$

Let LocFin be the category of T_1 spaces and continuous locally finite-to-one partial functions with closed domain. Proposition 6.4 constructs a functor

$$\begin{aligned} \mathrm{B}\colon \mathsf{LocFin} &\longrightarrow \mathsf{PreFinPf} \\ X &\longmapsto (X, \mathcal{B}_X) \\ f &\longmapsto f. \end{aligned}$$

Using the usual product of topological spaces, LocFin is a symmetric monoidal category. By Tychonoff's theorem, if $K \subseteq X$ and $K' \subseteq Y$ are compact, then so is $K \times K' \subseteq X \times Y$. Conversely, if $K \subseteq X \times Y$ is compact, it is included in $\pi_X(K) \times \pi_Y(K)$ where $\pi_X(K) \subseteq X$ and $\pi_Y(K) \subseteq Y$ are compact. This shows that our functor

$$\mathrm{B}\colon \mathsf{LocFin} \to \mathsf{PreFinPf}$$

is a strict symmetric monoidal functor.

Corollary 6.7. *An étale groupoid* $\mathcal{G}$

$$\begin{tikzcd} \mathcal{G}_1 \times_{\mathcal{G}_0} \mathcal{G}_1 \arrow[r, "m"] & \mathcal{G}_1 \arrow[loop above, "i"] \arrow[r, shift left=2, "d"] \arrow[r, shift right=2, "r"'] & \mathcal{G}_0 \arrow[l, "u" description] \end{tikzcd}$$

where $\mathcal{G}_1$ *is Hausdorff induces a semigroup* $(\mathcal{G}_1, \mathcal{B}_{\mathcal{G}_1}, m) \in \mathrm{SG}(\mathsf{PreFinPf})$*. Thus, it also induces a semigroup* $(\mathcal{G}_1, \mathcal{B}_{\mathcal{G}_1}^{\perp\perp}, m) \in \mathrm{SG}(\mathsf{FinPf})$*. Given a ring* $R \neq 0$*, the linearly Hausdorff* R*-algebra* $R\langle(\mathcal{G}_1, \mathcal{B}_{\mathcal{G}_1}, m)\rangle$ *has* $R\langle(\mathcal{G}_1, \mathcal{B}_{\mathcal{G}_1}^{\perp\perp}, m)\rangle$ *as completion and is complete if and only if* $(\mathcal{G}_1, \mathcal{B}_{\mathcal{G}_1})$ *is a finiteness space.*

Proof. It suffices to notice that m is a map $\mathcal{G}_1 \times \mathcal{G}_1 \to \mathcal{G}_1$ in LocFin. The domain $\mathcal{G}_1 \times_{\mathcal{G}_0} \mathcal{G}_1$ of m is closed since $\mathcal{G}_1$ is Hausdorff. The rest follows from Theorem 4.1. $\square$

In view of this corollary, a natural question is to find conditions for a topological space X to induce a finiteness space $(X, \mathcal{B}_X)$. We will need the following definitions.

Definition 6.8. Let X be a topological space.

- X is *hemicompact* if there exists an increasing chain of compact subsets $K_1 \subseteq K_2 \subseteq \cdots$ such that any compact subset K is contained in some K_i.
- X is *σ-compact* if it can be covered by a countable family of compact subsets. Note that every hemicompact space is σ-compact since singletons are compact.
- X is *σ-locally compact* if it is both σ-compact and locally compact.

Remark 6.9. Let X be a locally compact space. Then X is σ-compact if and only if it is hemicompact.

We now prove that hemicompact spaces (and in particular σ-locally compact spaces) induce finiteness spaces. This result originates from the work of the first, third and fourth authors.

Theorem 6.10. *If* X *is a hemicompact space, then* $(X, \mathcal{B}_X)$ *is a finiteness space.*

Proof. We already know $(X, \mathcal{B}_X)$ forms a pre-finiteness space. The rest follows immediately from Lemma 2.7. $\square$

This assumption of being hemicompact is by far not necessary.

Counterexample 6.11. Let X be an uncountable set endowed with the discrete topology. Then X is not hemicompact but $(X, \mathcal{B}_X)$ is a finiteness space.

We note also that being locally compact and Hausdorff is not enough.

Counterexample 6.12. It is well-known that the smallest uncountable ordinal ω_1, with the order topology, is locally compact and Hausdorff. Let us prove that $(\omega_1, \mathcal{B}_{\omega_1})$ is not a finiteness space. If K is a compact subset of ω_1, the family $\{[0, k+1) \mid k \in K\}$ is an open cover of K. Since it admits a finite subcover, K is countable by definition of ω_1. This shows that every bounded subset of ω_1 is countable. Conversely, if $A \subseteq \omega_1$ is countable, then it has an upper bound $\alpha < \omega_1$. To see this, we can choose $\alpha = \cup A$. It is a countable union of nested countable sets, hence is itself a countable ordinal. It follows that $A \subseteq [0, \alpha]$ and the latter is compact. We thus have

$$\mathcal{B}_{\omega_1} = \{u \subseteq \omega_1 \mid u \text{ is countable}\}.$$

An element of $\mathcal{B}_{\omega_1}^{\perp}$ cannot be infinite, since it would contain an infinite countable subset which is impossible by our description of $\mathcal{B}_{\omega_1}$. Therefore

$$\mathcal{B}_{\omega_1}^{\perp} = \{u' \subseteq \omega_1 \mid u' \text{ is finite}\}.$$

This implies that $\mathcal{B}_{\omega_1}^{\perp\perp} = \mathcal{P}(\omega_1)$ and $\mathcal{B}_{\omega_1} \neq \mathcal{B}_{\omega_1}^{\perp\perp}$ since ω_1 is uncountable.

7 An example

We now discuss an example introduced in [5]. Let $G = (V, E, d, r)$ be a directed graph.

$$E \underset{r}{\overset{d}{\rightrightarrows}} V$$

We assume that G is *row-finite*, i.e., that for all $v \in V$, we have $d^{-1}(v)$ is finite. We also assume that G is *countable*, i.e., E and V are countable sets. We let $F(G)$ be the set of all finite paths in G and $P(G)$ be the set of all infinite paths in G. If $\alpha \in F(G)$ and $\beta \in F(G)$ or $P(G)$ with $r(\alpha) = d(\beta)$, we let $\alpha\beta$ denote the evident path concatenation. We denote the *length* of α by $|\alpha|$. If $\alpha \in F(G)$, let

$$\mathcal{Z}(\alpha) = \{x \in P(G) \mid x = \alpha y \text{ for some } y \in P(G)\}.$$

Lemma 7.1 (See [5], Corollary 2.2)**.** *Let G be a countable row-finite directed graph. The family of sets*

$$\{\mathcal{Z}(\alpha) \mid \alpha \in F(G)\}$$

forms a basis of compact open sets for a σ-locally compact, totally disconnected, Hausdorff topology on $P(G)$, which coincides with the product topology obtained by viewing $P(G)$ as a subset of $\Pi_{i \in \mathbb{N}} E$, where E is given the discrete topology.

We can already note that as such, by Theorem 6.10, the space $\mathrm{B}(P(G)) = (P(G), \mathcal{B}_{P(G)})$ is a finiteness space.

Definition 7.2. Suppose $x, y \in P(G)$. We say that x and y are *shift equivalent with lag* $k \in \mathbb{Z}$ if there exists $N \in \mathbb{N}$ such that $N \geqslant -k$ and $x_i = y_{i+k}$ for all $i > N$. We write $x \sim_k y$ in this case.

Lemma 7.3. *We have $x \sim_0 x$ and $x \sim_k y \Rightarrow y \sim_{-k} x$ and $x \sim_k y, y \sim_l z \Rightarrow x \sim_{k+l} z$ for any $x, y, z \in P(G)$ and $k, l \in \mathbb{Z}$.*

One now defines a groupoid $\mathcal{G}$ as follows. Firstly, set $\mathcal{G}_0 = P(G)$ and

$$\mathcal{G}_1 = \{(x, k, y) \in P(G) \times \mathbb{Z} \times P(G) \,|\, x \sim_k y\}.$$

The domain and the range maps are given by $d(x, k, y) = x$ and $r(x, k, y) = y$ for any $(x, k, y) \in \mathcal{G}_1$ and the unit map is given, for any $x \in P(G)$, by $u(x) = (x, 0, x)$. The multiplication $m\colon \mathcal{G}_1 \times_{\mathcal{G}_0} \mathcal{G}_1 \to \mathcal{G}_1$ is defined as

$$m((x, k, y), (y, l, z)) = (x, k + l, z)$$

where

$$\mathcal{G}_1 \times_{\mathcal{G}_0} \mathcal{G}_1 = \{((x, k, y), (y, l, z)) \,|\, (x, k, y), (y, l, z) \in \mathcal{G}_1\}.$$

The inverse map is given by $i(x, k, y) = (y, -k, x)$ for $(x, k, y) \in \mathcal{G}_1$.

One now turns this groupoid into an étale groupoid. For $\alpha, \beta \in F(G)$ such that $r(\alpha) = r(\beta)$, one defines

$$\mathcal{Z}(\alpha, \beta) = \{(x, k, y) \in \mathcal{G}_1 \,|\, x \in \mathcal{Z}(\alpha),\ y \in \mathcal{Z}(\beta),\ k = |\beta| - |\alpha| \text{ and } x_i = y_{i+k} \ \ \forall i > |\alpha|\}.$$

Theorem 7.4 (See [5], Proposition 2.6)**.** *Let G be a countable row-finite directed graph. The family of sets*

$$\{\mathcal{Z}(\alpha, \beta) \,|\, \alpha, \beta \in F(G) \text{ and } r(\alpha) = r(\beta)\}$$

forms a basis of compact open sets for a second countable, σ-locally compact, Hausdorff topology on $\mathcal{G}_1$. With this topology on $\mathcal{G}_1$ and the topology on $\mathcal{G}_0$ described in Lemma 7.1, $\mathcal{G}$ is an étale groupoid.

Corollary 7.5. *Let G be a countable row-finite directed graph. Then $(\mathcal{G}_1, \mathcal{B}_{\mathcal{G}_1}, m)$ is a semigroup in* FinPf *where $\mathcal{G}_1$ and m are as above.*

Proof. This follows immediately from Corollary 6.7 and Theorems 6.10 and 7.4. $\square$

References

[1] R. Blute, Hopf algebras and linear logic, *Mathematical Structures in Computer Science* **6**, pp. 189–217, (1996).

[2] R. Blute, J.R.B. Cockett, P.-A. Jacqmin, P. Scott, Finiteness spaces and generalized power series, *Electronic Notes in Theoretical Computer Science* **341**, pp. 5–22, (2018).

[3] T. Ehrhard, Finiteness spaces, *Mathematical Structures in Computer Science* **15**, pp. 615–646, (2005).

[4] I.M. James, Introduction to uniform spaces, *London Mathematical Society Lecture Note Series* **144**, (1990).

[5] A. Kumjian, D. Pask, I. Raeburn, J. Renault, Graphs, groupoids and Cuntz-Krieger algebras, *Journal of Functional Analysis* **144**, pp. 505–541, (1997).

[6] S. Lefschetz, Algebraic topology, *American Mathematical Society Colloquium Publications* **27**, (1942).

[7] J. Renault, A groupoid approach to $C^\star$-algebras, *Lecture Notes in Mathematics, Springer-Verlag* **793**, (1980).

[8] P. Ribenboim, Noetherian rings of generalized power series, *Journal of Pure and Applied Algebra* **79**, pp. 293–312, (1992).

[9] P. Ribenboim, Rings of generalized power series: II: Units and zero-divisors, *Journal of Algebra* **168**, pp. 71–89, (1994).

[10] I.G. Rosenberg, Algebraic properties of a general convolution, *London Mathematical Society Lecture Note Series* **131**, pp. 175–204, (1988).

[11] A. Sims, Hausdorff étale groupoids and their $C^\star$-algebras, *arXiv:1710.10897*, (2017).